

# THE ANALYTICAL SOLUTION OF THE PROBLEM ON PLASMA OSCILLATIONS IN HALF-SPACE WITH SPECULAR BOUNDARY CONDITIONS

© 2017 A. V. Latyshev, S. Sh. Suleymanova

*Faculty of Physics and Mathematics,*

*Moscow State Regional University*

*105005, Moscow, Radio str., 10-A*

*e-mail: avlatyshev@mail.ru, sevda-s@yandex.ru*

The boundary problem about behavior (oscillations) of the electronic plasmas with arbitrary degree of degeneration of electronic gas in half-space with specular boundary conditions is analytically solved. The kinetic equation of Vlasov–Boltzmann with integral of collisions of type BGK (Bhatnagar, Gross, Krook) and Maxwell equation for electric field are applied. Distribution function for electrons and electric field in plasma in the form of expansion under eigen solutions of the initial system of equations are received. Coefficients of these expansions are found by means of the boundary conditions.

**Keywords:** Vlasov—Boltzmann equation, Maxwell equation, frequency of collisions, electromagnetic field, modes of Drude, Debaye, and Van Kampen, dispersion function, boundary value Riemann problem.

## INTRODUCTION

The present paper is a continuation of a series papers devoted to the problem of behavior non-degenerate and degenerate (Maxwell) plasma in a half-space, which is set on the border of the external longitudinal electric field (see, for example, [5] – [8]). In the works [5] and [6] considered the case of a degenerate plasma. In the works [7] and [8] considered the case of a non-degenerate Maxwell plasma. In this paper we consider the general case of a plasma with an arbitrary degree of degeneration of the electronic gas.

The concept of "plasma" appeared in the works of Tonks and Langmuir for the first time [14]. The problem of electron plasma oscillations was considered by A.A. Vlasov (see, for example, [18]). The problem of plasma oscillation turns out to be formulated correctly as a boundary problem of mathematical physics in the work [4].

In this paper, the kinetic equation of Vlasov–Boltzmann with integral of collisions of type BGK (Bhatnagar, Gross, Krook) and Maxwell equation for electric field are applied for solution the problem of the behavior of the plasma with an arbitrary degree of degeneration of the electronic gas in a half-space in an external alternating longitudinal electric field.

Principal step is the reduction of the boundary problem for one-dimensional and one-velocity. We use the method of successive approximations, linearization of equations with respect to the absolute Fermi-Dirac distribution of electrons and the law of conservation of the number of particles. Then, separation of variables reduces equations of the problem

to the characteristic system of the equations. In the space of generalized functions are their eigen solutions of the original system corresponding the continuous spectrum (mode of Van Kampen).

By solving the dispersion equation, we find their eigen solutions, which are responsible accession and the discrete spectrum (modes of Drude and Debye). Then the common decision of a boundary problem in the form of expansion of eigen solutions is formed. The expansion coefficients are determined from the boundary conditions. This allows for expansion of the distribution function and the electric field explicitly.

It determines the structure of the screened electric field. It turned out that there is a domain of values parameters of the problem, in which there is no mode of Debye.

It is shown firstly, mode of Drude describing the volume conductivity, exists at all values of the parameters of the problem. Secondly, mode of Debye describes screening the electric field and exists for frequencies of oscillation in the external field which less a some critical frequency (it is located near the plasma resonance). And mode Van Kampen [15] representing a mix of (chaotization) eigen solutions of the Vlasov–Boltzmann equation also exist for all values of the parameters of the problem.

## 1. PROBLEM STATEMENT AND THE BASIC EQUATIONS

For analytical solution such difficult problem as the problem of plasma oscillations, it must first be reduced to one-dimensional and one-velocity [10]. To do this, the vector of the external electric field is directed along the same axis, which is orthogonal to the surface layer with the plasma.

Further we pass to the dimensionless variables and parameters. If dimensionless equations naturally arises small parameter such as a perturbation of the dimensionless (chemical) potential caused by the presence of an external electric field. Using the method of small parameter, we linearize the problem relative to the absolute Fermi–Dirac distribution of the electron. Applying the law of conservation of the number of particles, we finally formulate the problem in the form of one-dimensional and one-velocity boundary value problem

$$\mu \frac{\partial H}{\partial x_1} + w_0 H(x, \mu) = \mu e(x_1) + \int_{-\infty}^{\infty} k(\mu', \alpha) H(x, \mu') d\mu', \quad (1.1)$$

$$\frac{de(x_1)}{dx_1} = \varkappa^2(\alpha) \int_{-\infty}^{\infty} k(\mu', \alpha) H(x_1, \mu') d\mu'. \quad (1.2)$$

In the equations (1.1) and (1.2)

$$\varkappa^2(\alpha) = \frac{32\pi^2 e^2 p_T^3 s_0(\alpha)}{(2\pi\hbar)^3 m \nu^2}, \quad w_0 = 1 - i \frac{\omega}{\nu} = 1 - i\omega\tau = 1 - i \frac{\Omega}{\varepsilon},$$

where  $\Omega = \omega/\omega_p$ ,  $\varepsilon = \nu/\omega_p$ ,  $\omega_p$  is plasma (Langmuir) frequency,  $\omega_p = \sqrt{4\pi e^2 N/m}$ . Here  $N$  is numerical density (concentration) of electrons in the equilibrium state.

Let's express parameter  $\varkappa$  through numerical density. From definition of a numerical density it follows that

$$N = \int f_0(P, \alpha) d\Omega_F = \frac{2p_T^3}{(2\pi\hbar)^3} \int \frac{d^3P}{1 + e^{P^2 - \alpha}} = \frac{8\pi p_T^3}{(2\pi\hbar)^3} \int_0^\infty \frac{P^2 dP}{1 + e^{P^2 - \alpha}} = \frac{8\pi p_T^3}{(2\pi\hbar)^3} s_2(\alpha),$$

where

$$s_2(\alpha) = \int_0^\infty \frac{P^2 dP}{1 + e^{P^2 - \alpha}} = \frac{1}{2} l_0(\alpha), \quad l_0(\alpha) = \int_0^\infty \ln(1 + e^{\alpha - P^2}) dP.$$

Hence, the numerical density of particles of plasma and thermal wave number  $k_T = mv_T/\hbar$  are related

$$N = \frac{l_0(\alpha)}{2\pi^2} k_T^3 = \frac{s_2(\alpha)}{\pi^2} k_T^3.$$

Not difficult obtain that

$$\varkappa^2(\alpha) = \frac{\omega_p^2}{\nu^2} \cdot \frac{s_0(\alpha)}{s_2(\alpha)} = \frac{\omega_p^2}{\nu^2} \cdot \frac{1}{r(\alpha)} = \frac{1}{\varepsilon^2 r(\alpha)},$$

where

$$r(\alpha) = \frac{s_2(\alpha)}{s_0(\alpha)}, \quad \varepsilon = \frac{\nu}{\omega_p}.$$

It is known that the frequency of the plasma oscillations are usually much more than the frequency of collisions between electrons in the metal [11]. Therefore, in the case when  $\omega \sim \omega_p$  the condition  $\omega_p \gg \nu$  is performed.

Consider the condition of specular reflection of electrons from the boundary of half-space

$$f(x=0, v_x, v_y, v_z, t) = f_{eq}(x=0, -v_x, v_y, v_z, t), \quad v_x > 0.$$

We will linearize this boundary condition, we obtain

$$H(0, \mu) = H(0, -\mu), \quad 0 < \mu < 1. \quad (1.3)$$

The boundary condition for the field on the surface of the plasma has the form

$$e(0) = 1, \quad (1.4)$$

and away from the surface of the field intended to be limited

$$e(+\infty) = e_\infty, \quad |e_\infty| < +\infty. \quad (1.5)$$

We need the condition of non-permeability of electrons through the plasma boundary as a boundary condition

$$\int v_x f(x, \mathbf{v}, t) d\Omega_F = 0.$$

Hence we obtain the following integral condition

$$\int_{-\infty}^{\infty} \mu' H(0, \mu') f_0(\mu', \alpha) d\mu' = 0. \quad (1.6)$$

Condition (1.6) is the condition of non-permeability of electrons through the plasma boundary.

## 2. EIGENFUNCTIONS OF THE CONTINUOUS SPECTRUM

First, we seek the general solution of the system of equations (1.1) and (1.2).

Application of the general Fourier method of the separation of variables in several steps results in the following substitution

$$H_\eta(x, \mu) = \exp\left(-\frac{w_0 x}{\eta}\right) \Phi(\eta, \mu), \quad e_\eta(x) = \exp\left(-\frac{w_0 x}{\eta}\right) E(\eta), \quad (2.1)$$

where  $\eta$  is the spectrum parameter or the parameter of separation which is complex in general.

We substitute the equalities (2.1) into the equations (1.1) and (1.2). We obtain the following characteristic system of equations

$$(\eta - \mu)\Phi(\eta, \mu) = \eta\mu \frac{E(\eta)}{w_0} + \frac{\eta}{w_0} \int_{-\infty}^{\infty} k(\mu', \alpha)\Phi(\eta, \mu')d\mu', \quad (2.2)$$

$$-\frac{w_0}{\eta}E(\eta) = \frac{1}{\varepsilon^2 r(\alpha)} \int_{-\infty}^{\infty} k(\mu', \alpha)\Phi(\eta, \mu')d\mu'. \quad (2.3)$$

Let us introduce the designations

$$n(\eta) = \int_{-\infty}^{\infty} k(\mu', \alpha)\Phi(\eta, \mu')d\mu'.$$

By means of this equality we will rewrite the equations (2.2) and (2.3) in the form

$$(\eta - \mu)\Phi(\eta, \mu) = \frac{E(\eta)}{w_0} \mu\eta + \frac{\eta n(\eta)}{w_0}, \quad (2.4)$$

$$-\frac{w_0}{\eta}E(\eta) = \frac{n(\eta)}{\varepsilon^2 r(\alpha)}. \quad (2.5)$$

Let us introduce the designations

$$\eta_1^2 \equiv \eta_1(\alpha) = w_0 \varepsilon^2 \frac{s_2(\alpha)}{s_0(\alpha)} = w_0 \varepsilon^2 r(\alpha) = \varepsilon(\varepsilon - i\Omega)r(\alpha).$$

From equations (2.4) and (2.5) we obtain the following equation

$$(\eta - \mu)\Phi(\eta, \mu) = \frac{E(\eta)}{w_0} (\eta\mu - \eta_1^2). \quad (2.6)$$

For  $\eta \in (-\infty, +\infty)$  we look for a solution of equation (2.6) in the space of generalized function [17]

$$\Phi(\eta, \mu) = \frac{E(\eta)}{w_0} (\mu\eta - \eta_1^2) P \frac{1}{\eta - \mu} + g(\eta) \delta(\eta - \mu). \quad (2.7)$$

In the equation (2.7)  $\eta \in (-\infty, +\infty)$ ,  $\mu \in (-\infty, +\infty)$ . The set of values  $\eta$ , filling the real line  $-\infty < \eta < +\infty$  is called a continuous spectrum of the characteristic equation.

In the equation (2.7)  $\delta(x)$  is the Dirac delta function, the symbol  $Px^{-1}$  denotes the distribution, i.e. the principal value of the integral of  $x^{-1}$ , function  $g(\eta)$  acts as an arbitrary "constant" of integration.

The solution (2.7) of the equation (2.6) are called the eigenfunctions of the characteristic equation.

To find the function  $g(\eta)$ , we substitute (2.7) in the definition of normalization functions  $n(\eta)$ . The result is that  $g(\eta) = \eta_1^2 E(\eta) \Lambda(\eta) / [\eta k(\eta, \alpha)]$ . Here the dispersion function is entered

$$\Lambda(z) = \Lambda(z, \Omega, \varepsilon, \alpha) = 1 + \frac{z}{w_0 \eta_1^2} \int_{-\infty}^{\infty} \frac{\eta_1^2 - \mu' z}{\mu' - z} k(\mu', \alpha) d\mu'. \quad (2.8)$$

Eigenfunctions (2.7) of the characteristic equation (2.6) using (2.8) can be represented as

$$\Phi(\eta, \mu) = \frac{E(\eta)}{w_0} \left[ P \frac{\mu \eta - \eta_1^2}{\eta - \mu} - w_0 \eta_1^2 \frac{\Lambda(\eta)}{\eta k(\eta, \alpha)} \delta(\eta - \mu) \right]. \quad (2.9)$$

The collection of eigenfunctions  $\Phi(\eta, \mu)$  of the characteristic equation corresponds to the continuous spectrum. They are often called "mode Van Kampen" (see [15] and [3]).

Eigenfunction can be represented as

$$\Phi(\eta, \mu) = \frac{E(\eta)}{w_0} F(\eta, \mu),$$

where

$$F(\eta, \mu) = P \frac{\mu \eta - \eta_1^2}{\eta - \mu} - w_0 \eta_1^2 \frac{\Lambda(\eta)}{\eta k(\eta, \alpha)} \delta(\eta - \mu).$$

The dispersion function problems  $\Lambda(z)$  can be represented in the following form

$$\Lambda(z) = 1 - \frac{1}{w_0} - \frac{z^2 - \eta_1^2}{w_0 \eta_1^2} \lambda_0(z, \alpha).$$

Here function is entered

$$\lambda_0(z, \alpha) = 1 + z \int_{-\infty}^{\infty} \frac{k(\mu, \alpha) d\mu}{\mu - z}.$$

For its boundary values above and below are carried out on the real axis of the Sokhotzky formulas [2]  $\lambda_0^\pm(\mu, \alpha) = \lambda_0(\mu, \alpha) \pm i\pi \mu k(\mu, \alpha)$ .

Using these formulas, we can easily calculate the boundary values of above and below on the real axis dispersion function of problems

$$\Lambda^\pm(\mu) = \Lambda(\mu) \pm i \frac{\pi}{w_0 \eta_1^2} \mu k(\mu, \alpha) (\eta_1^2 - \mu^2), \quad \frac{\Lambda^+(\mu) + \Lambda^-(\mu)}{2} = \Lambda(\mu).$$

Fig. 1 and 2 are graphs respectively the real and imaginary parts of the dispersion function  $\Lambda^+(\mu)$  in the case  $\Omega = 1$  and  $\varepsilon = 0.01$ , curves 1, 2, 3 correspond to the values of the dimensionless chemical potential  $\alpha = 3, 0, -1$ .

### 3. ZEROS DISPERSION FUNCTION

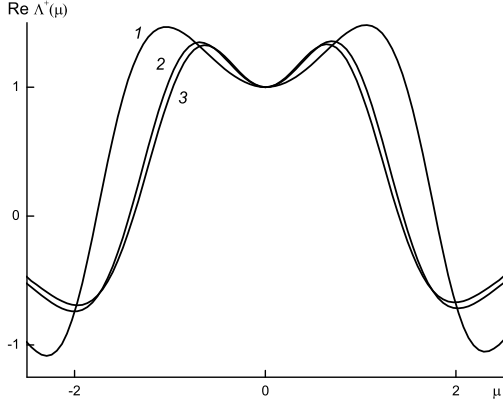


Fig. 1.

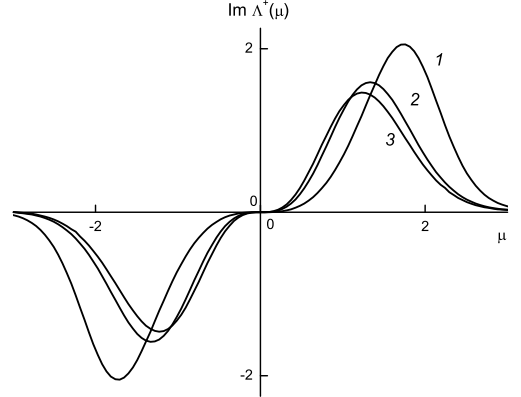


Fig. 2.

We find the zeros of the dispersion equation

$$\frac{\Lambda(z)}{z} = 0. \quad (3.1)$$

It is easy to see that the value of the dispersion function at infinity is equal to

$$\Lambda_\infty = \Lambda(\infty) = 1 - \frac{1}{w_0} + \frac{1}{w_0^2 \varepsilon^2}.$$

Hence we find that

$$\Lambda_\infty = \frac{-i\nu\omega + \omega_p^2 - \omega^2}{(\nu - i\omega)^2} \neq 0$$

for any  $\nu \neq 0$ , i.e. in any collisional plasma.

This means that the point  $z_i = \infty$  is a zero dispersion equation. This point is point of the spectrum associated to the continuous spectrum. Point  $z_i = \infty$  corresponds to the following solution of the original system of equations (2.5) and (2.6)

$$H_\infty(x, \mu) = \frac{E_\infty}{w_0} \cdot \mu, \quad e_\infty = E_\infty. \quad (3.2)$$

Here  $E_\infty$  is an arbitrary constant.

The solution (3.2) does not depend on the chemical potential. This solution is naturally called as mode of Drude. It describes the volume conductivity of metal, considered by Drude (see, for example, [1]).

By definition, the discrete spectrum of the characteristic equation (2.6) is the set of finite complex zeros of the dispersion equation (3.1) which do not lie on the real axis (a cut dispersion function).

We start to search zeros of the dispersion function. Let us take Laurent series of the dispersion function

$$\Lambda(z) = \Lambda_\infty + \frac{\Lambda_2}{z^2} + \frac{\Lambda_4}{z^4} + \dots, \quad z \rightarrow \infty. \quad (3.3)$$

where

$$\Lambda_2 = \frac{s_4(\alpha) - \eta_1^2 s_2(\alpha)}{w_0 \eta_1^2 s_0(\alpha)}, \quad \Lambda_4 = \frac{s_6(\alpha) - \eta_1^2 s_4(\alpha)}{w_0 \eta_1^2 s_0(\alpha)}, \dots, \quad s_n(\alpha) = \int_0^\infty \mu^n f_0(\mu, \alpha) d\mu.$$

From the expansion (3.3) we see that in a neighborhood of infinity, there are two zeros  $\pm \eta_0$  dispersion function  $\Lambda(z)$

$$\pm \eta_0 \approx \sqrt{-\frac{\Lambda_2}{\Lambda_\infty}}. \quad (3.4)$$

Since the dispersion function is even then its zeros differ from each other by sign. By zero  $\eta_0$  we understand such radical value from (3.4) that  $\text{Re}(w_0/\eta_0) > 0$ . For such a zero exponent  $\exp[-(w_0/\eta_0)x]$  is monotonically decreasing as  $x \rightarrow +\infty$ .

Zero  $\eta_0$  corresponds to the following equation

$$H_{\eta_0}(x, \mu) = \exp\left(-\frac{w_0}{\eta_0}x\right) \Phi(\eta_0, \mu), \quad e_{\eta_0}(x) = \exp\left(-\frac{w_0}{\eta_0}x\right) E_0.$$

Here

$$\Phi(\eta_0, \mu) = \frac{E_0}{w_0} \frac{\eta_0 \mu - \eta_1^2}{\eta_0 - \mu}.$$

This solution is naturally called the mode of Debye (this is plasma mode). In the case of low frequencies it describes well-known screening of Debay [1].

Equality (3.4) we will present in the explicit form

$$\eta_0 = \eta_0(\alpha, \Omega, \varepsilon) \approx \sqrt{\frac{(\Omega + i\varepsilon)^2 [\eta_1^2 s_2(\alpha) - s_4(\alpha)]}{w_0 \eta_1^2 s_0(\alpha) (\Omega^2 - 1 + i\varepsilon \Omega)}}. \quad (3.5)$$

From (3.5) we see that near the plasma resonance (when  $\Omega \approx 1$ , i.e.  $\omega \approx \omega_p$ ) module of zero  $|\eta_0(\alpha, \Omega, \varepsilon)|$  becomes unlimited for all values of the dimensionless chemical potential  $\alpha$  in the case  $\varepsilon \rightarrow 0$ .

#### 4. ON THE EXISTENCE OF PLASMA MODES

Zero  $\eta_0$  is a function of the parameters of the initial system of equations  $\mu$ ,  $\omega$  and  $\nu$  or the function of the parameters  $(\alpha, \Omega, \varepsilon)$ . Required to find the domain  $D^+(\alpha)$ , which lies in the plane of the parameters  $(\Omega, \varepsilon)$ , such that if  $(\Omega, \varepsilon) \in D^+(\alpha)$ , then the number of zeros  $N$  of the dispersion function  $\Lambda(z)$  is two  $N = 2$ . The  $D^-(\alpha)$  denotes a domain in the plane of the parameters that the number of zeros of the dispersion function is zero  $N = 0$ . The curve, which is the boundary of these domains, denoted by  $L = L(\alpha)$ .

The set of physically significant parameters  $(\Omega, \varepsilon)$  fills a quarter-plane  $\mathbb{R}_+^2 = \{(\Omega, \varepsilon) : \Omega \geq 0, \varepsilon \geq 0\}$ . Case  $\Omega \geq 0$  (or  $\omega = 0$ ) corresponds to the external stationary electric field, and case  $\varepsilon = 0$  (or  $\nu = 0$ ) corresponds to the case is responsible collision-less plasma.

We take the contour  $\Gamma_\rho = C_\rho^+ \cup C_\rho^-$ . This contour consists of two closed semi-circles  $C_\rho^+$  and  $C_\rho^-$  of radius  $R = 1/\rho$  lying in the upper and lower half-planes;  $\rho$  is sufficiently small positive real number,  $C_\rho^\pm = \{z = x + iy, |z| = 1/\rho, |x \pm i\rho| \leq 1/\rho\}$ . The number  $R$

we take large enough to zero of dispersion function (if they exist) lying inside the domain  $D_\rho$  bounded by the contour  $\Gamma_\rho$ . Note that if  $\rho \rightarrow 0$  domain  $D_\rho$  passes to  $D_0$  bounded by the contour  $\Gamma_0 = \lim_{\rho \rightarrow 0} \Gamma_\rho$ . This domain coincides with the complex plane with a cut along the real axis.

Then according to the principle of argument the number [2, 13] of zeros  $N$  of the dispersion function in the domain  $D_\rho$  equals to

$$N = \frac{1}{2\pi i} \oint_{\Gamma_\rho} d \ln \Lambda(z).$$

Considering the limit in this equality when  $\rho \rightarrow 0$  and taking into account that the dispersion function is analytic in the neighborhood of the infinity, we obtain that

$$N = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d \ln \Lambda^+(\tau) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} d \ln \Lambda^-(\tau).$$

After some transformations, we obtain that

$$N = \frac{1}{\pi i} \int_0^{\infty} d \ln \frac{\Lambda^+(\tau)}{\Lambda^-(\tau)} = 2\kappa_{\mathbb{R}_+}(G). \quad (4.1)$$

Here  $\kappa_{\mathbb{R}_+}(G)$  is the index of the function  $G(\tau) = \Lambda^+(\tau)/\Lambda^-(\tau)$ , calculated along the positive real axis.

Thus, equality (4.1) means that the number of zeros of the dispersion function  $\Lambda(z)$  is doubled the index function  $G(\tau)$  calculated along the positive real axis.

Consider a curve in the complex plane  $\Gamma_\alpha = \Gamma(\alpha)$ ,

$$\Gamma(\alpha) : z = G(\tau), \quad 0 \leq \tau \leq +\infty,$$

It is obvious that  $G(0) = 1$ ,  $\lim_{\tau \rightarrow +\infty} G(\tau) = 1$ . Consequently, according to (4.1), the number of values  $N$  equals to doubled number of turns of the curve  $\Gamma(\alpha)$  around the point of origin, i.e.

$$N = 2\kappa(G), \quad \kappa(G) = \text{ind}_{[0, +\infty]} G(\tau).$$

Let us single real and imaginary parts of the function  $G(\mu)$  out. At first, we represent the function  $G(\mu)$  in the form

$$G(\mu) = \frac{\Omega^+(\mu)}{\Omega^-(\mu)},$$

where

$$\Omega^\pm(\mu) = (w_0 - 1)\eta_1^2 + (\eta_1^2 - \mu^2)\lambda_0(\mu, \alpha) \pm is(\mu, \alpha)(\eta_1^2 - \mu^2), \quad s(\mu, \alpha) = \frac{\pi}{2s_0(\alpha)}\mu f_0(\mu, \alpha).$$

Taking into account that

$$w_0 - 1 = -i\frac{\Omega}{\varepsilon}, \quad \eta_1^2 = \varepsilon r(\alpha)(\varepsilon - i\Omega), \quad (w_0 - 1)\eta_1^2 = -\Omega(\Omega + i\varepsilon)r(\alpha).$$

Single out the the real and imaginary parts of the functions  $\Omega^\pm(\mu)$ . We have

$$\Omega^\pm(\mu) = -P^\pm(\mu) - iQ^\pm(\mu),$$



where

$$P^\pm(\mu) = (1 + \gamma)^2 r(\alpha) + \lambda_0(\mu, \alpha)(\mu^2 - \varepsilon^2 r(\alpha)) \mp \varepsilon(1 + \gamma)r(\alpha)s(\mu, \alpha),$$

$$Q^\pm(\mu) = \varepsilon(1 + \gamma)r(\alpha)(1 + \lambda_0(\mu, \alpha)) \pm (\mu^2 - \varepsilon^2 r(\alpha))s(\mu, \alpha).$$

Now the coefficient  $G(\mu)$  can be represented as

$$G(\mu) = \frac{P^+(\mu) + iQ^+(\mu)}{P^-(\mu) + iQ^-(\mu)}.$$

We can easily single real and imaginary parts of the function  $G(\mu)$  out

$$G(\mu, \alpha) = \frac{P^+P^- + Q^+Q^-}{(P^-)^2 + (Q^-)^2} + i \frac{P^-Q^+ - P^+Q^-}{(P^-)^2 + (Q^-)^2},$$

or briefly

$$G(\mu) = G_1(\mu) + iG_2(\mu),$$

where

$$G_1(\mu) = \frac{g_1(\mu)}{g(\mu)}, \quad G_2(\mu) = \frac{g_2(\mu)}{g(\mu)}.$$

Here

$$\begin{aligned} g(\mu) &= [P^-(\mu)]^2 + [Q^-(\mu)]^2 = [\Omega^2 r(\alpha) + \lambda_0(\mu, \alpha)(\mu^2 - \varepsilon^2 r(\alpha)) + \varepsilon \Omega r(\alpha)s(\mu, \alpha)]^2 + \\ &\quad + [\varepsilon \Omega (1 + \lambda_0(\mu, \alpha)) - s(\mu, \alpha)(\mu^2 - \varepsilon^2 r(\alpha))]^2, \end{aligned}$$

$$\begin{aligned} g_1(\mu) &= P^+(\mu)P^-(\mu) + Q^+(\mu)Q^-(\mu) = [\Omega^2 r(\alpha) + \lambda_0(\mu, \alpha)(\mu^2 - \varepsilon^2 r(\alpha))]^2 + \\ &\quad + \varepsilon^2 \Omega^2 r^2(\alpha)[(1 + \lambda_0(\mu, \alpha))^2 - s^2(\mu, \alpha)] - (\mu^2 - \varepsilon^2 r(\alpha))^2 s^2(\mu, \alpha), \end{aligned}$$

$$\begin{aligned} g_2(\mu) &= P^-(\mu)Q^+(\mu) - P^+(\mu)Q^-(\mu) = 2s(\mu, \alpha) \{ [\Omega^2 r(\alpha) + \\ &\quad + \lambda_0(\mu, \alpha)(\mu^2 - \varepsilon^2 r(\alpha))](\mu^2 - \varepsilon^2 r(\alpha)) + \varepsilon^2 \Omega^2 r^2(\alpha)(1 + \lambda_0(\mu, \alpha)) \}. \end{aligned}$$

We consider (see fig. 3,4) the curve  $L_\alpha = L(\alpha, \Omega, \varepsilon)$  which is defined in implicit form by the following parametric equations

$$L_\alpha = L_\alpha(\Omega, \varepsilon) : \quad g_1(\mu, \alpha, \Omega, \varepsilon) = 0, \quad g_2(\mu, \alpha, \Omega, \varepsilon) = 0, \quad 0 \leq \mu \leq +\infty,$$

and which lays in the plane of the parameters of the problem  $(\Omega, \varepsilon)$ , and when passing through this curve the index of the function  $G(\mu)$  at the positive semi-axis changes stepwise.

Each curve  $L_\alpha$  separates the plane of the parameters  $(\Omega, \varepsilon)$  into two domains  $D^+(\alpha)$  and  $D^-(\alpha)$ , such that if the point  $(\Omega, \varepsilon)$  passes from one domain to another the index of the function  $G(\mu)$  at the positive semi-axis changes stepwise.

Fig. 3 and 4 is a graph of the curve  $L$  which separates the domain  $D^+$  and  $D^-$  with the corresponding values of the dimensionless chemical potential  $\alpha = -3, 3$ .

As in the work [9] we can prove that if  $(\Omega, \varepsilon) \in D^+(\alpha)$  then  $\varkappa_{[0,+\infty]}(G) = 1$  (the curve  $L$  encircles the point of origin once), and if  $(\Omega, \varepsilon) \in D^-(\alpha)$  then  $\varkappa_{[0,+\infty]}(G) = 0$  (the curve  $L$  does not encircle the point of origin).

From the expression (4.1) one can see that the number of zeros of the dispersion function is two ( $N = 2$ ), if  $(\Omega, \varepsilon) \in D^+(\alpha)$  and the dispersion function does not have zeros, if  $(\Omega, \varepsilon) \in D^-(\alpha)$ .

We note, that in the work [9] the method of analysis of boundary regime when  $(\Omega, \varepsilon) \in L_\alpha$  was developed.

We deduce explicit parametric equations of the curve  $L_\alpha$  which separated quarter-plane of the parameters  $(\Omega, \varepsilon)$  in the two domains  $D^+(\alpha)$  and  $D^-(\alpha)$ .

From the equation  $g_2(\mu, \alpha, \Omega, \varepsilon) = 0$  we find

$$\Omega^2 = -\frac{1}{r(\alpha)} \cdot \frac{(\mu^2 - \varepsilon^2 r(\alpha))\lambda_0(\mu, \alpha)}{\mu^2 + \varepsilon^2 r(\alpha)\lambda_0(\mu, \alpha)}. \quad (4.2)$$

Consider the equation  $g_1(\mu, \alpha, \Omega, \varepsilon) = 0$ . Let us transform this equation with the help of the equation (4.2). We perform this calculation in general. From the equation  $g_1 = P^-Q^+ - P^+Q^- = 0$  we find  $P^- = P^+(Q^-/Q^+)$ . Further, we find that

$$g_1 \Big|_{g_2=0} = [P^-Q^+ - P^+Q^-] \Big|_{P^- = \frac{Q^-}{Q^+}P^+} = \frac{Q^-}{Q^+} [(P^+)^2 + (P^-)^2].$$

It is obvious that the equation  $g_1 \Big|_{g_2=0} = 0$  is equivalent to the equation  $Q^-(\mu) = 0$ . From this equation and (4.2) we find the parametric equations of curves  $L_\alpha$ :

$$L_\alpha : \quad \Omega = \sqrt{L_1(\mu)}, \quad \varepsilon = \sqrt{L_2(\mu)}, \quad 0 \leq \mu \leq +\infty. \quad (4.3)$$

In (4.3) we introduce the designations

$$L_1(\mu) = \frac{s_0(\alpha)}{s_2(\alpha)} \cdot \frac{\mu^2[\lambda_0(\mu, \alpha)(1 + \lambda_0(\mu, \alpha)) + s^2(\mu, \alpha)]^2}{[-\lambda_0(\mu, \alpha)][(1 + \lambda_0(\mu, \alpha))^2 + s^2(\mu, \alpha)]}$$

and

$$L_2(\mu) = \frac{s_0(\alpha)}{s_2(\alpha)} \cdot \frac{\mu^2 s_2(\mu, \alpha)}{[-\lambda_0(\mu, \alpha)][(1 + \lambda_0(\mu, \alpha))^2 + s^2(\mu, \alpha)]}.$$

Thus, we have constructed a curve  $L_\alpha$  which is the boundary of domains  $D^+(\alpha)$  and  $D^-(\alpha)$ . Recall that if  $(\Omega, \varepsilon) \in D^+(\alpha)$  then

$$\varkappa(G) = \text{ind}_{[0,+\infty]} \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} = 1.$$

This means that the curve  $\Gamma_\alpha$  encircles the point of origin once. And if  $(\Omega, \varepsilon) \in D^-(\alpha)$  then

$$\varkappa(G) = \text{ind}_{[0,+\infty]} \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} = 0.$$

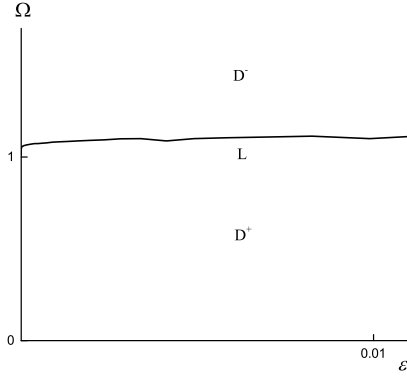


Fig. 3.

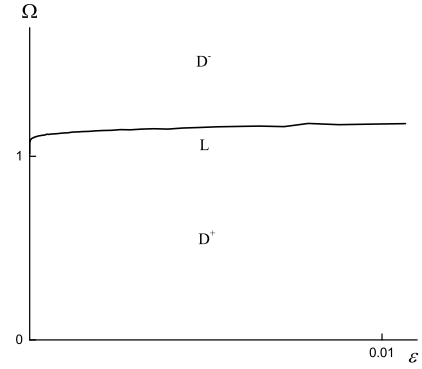


Fig. 4.

This means that the curve  $\Gamma_\alpha$  does not encircle the point of origin. The curve  $\Gamma(\alpha)$  in the complex plane  $\mathbb{C}$  is determined by the equations

$$\Gamma_\alpha : \quad x = \operatorname{Re} \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)}, \quad y = \operatorname{Im} \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)}, \quad 0 \leq \mu \leq +\infty.$$

We accentuate that the curve  $L_\alpha$  on the plane of the parameters  $(\Omega, \varepsilon)$  defined by the parametric equations

$$L(\alpha) : \quad \operatorname{Re} \frac{\Lambda^+(\mu, \alpha, \Omega, \varepsilon)}{\Lambda^-(\mu, \alpha, \Omega, \varepsilon)} = 0, \quad \operatorname{Im} \frac{\Lambda^+(\mu, \alpha, \Omega, \varepsilon)}{\Lambda^-(\mu, \alpha, \Omega, \varepsilon)} = 0, \quad 0 \leq \mu \leq +\infty.$$

The value of the reduced chemical potential  $\alpha$  fills the entire real line  $-\infty \leq \alpha \leq +\infty$ . In this case  $\alpha = -\infty$  corresponds to a Maxwell plasma and the case  $\alpha = +\infty$  corresponds to a completely degenerate plasma.

We formulate conclusions in terms of the plasma (Debye or discrete) mode. If  $(\Omega, \varepsilon) \in D^+(\alpha)$  then plasma mode  $H_{\eta_0}(x_1, \mu), e_{\eta_0}(x_1)$  exists (the number of zeros of the dispersion function  $\Lambda(z)$  is equal to two or index of coefficient  $G(\mu) = \Lambda^+(\mu)/\Lambda^-(\mu)$  is equal to one in the real semi-axis). If  $(\Omega, \varepsilon) \in D^-(\alpha)$  then plasma mode does not exist (the number of zeros of the dispersion function is zero or the index of the coefficient is equal to zero on the real semi-axis).

## 5. SPECULAR REFLECTION OF ELECTRONS FROM THE PLASMA BOUNDARY

We will solve the problem which consists of equations (1.1) and (1.2) with the boundary conditions (1.3) - (1.6). We look for the solution of a problem in the form of expansions

$$H(x, \mu) = \frac{E_\infty}{w_0} \mu + \frac{E_0}{w_0} \frac{\eta_0 \mu - \eta_1^2}{\eta_0 - \mu} \exp\left(-\frac{w_0 x}{\eta_0}\right) + \frac{1}{w_0} \int_0^\infty \exp\left(-\frac{w_0 x}{\eta}\right) F(\eta, \mu) E(\eta) d\eta, \quad (5.1)$$

$$e(x) = E_\infty + E_0 \exp\left(-\frac{w_0 x}{\eta_0}\right) + \int_0^\infty \exp\left(-\frac{w_0 x}{\eta}\right) E(\eta) d\eta. \quad (5.2)$$

The unknowns in the expansions (5.1) and (5.2) are the coefficients discrete spectrum  $E_0$ ,  $E_\infty$  and the coefficient of the continuous spectrum  $E(\eta)$  and if  $(\Omega, \varepsilon) \in D^-(\alpha)$  then  $E_0 = 0$ .

Consider the case  $(\Omega, \varepsilon) \in D^+(\alpha)$ . We substitute the expansions (5.1) and (5.2) into the boundary conditions (1.3) - (1.6). We obtain the following system of equations

$$2E_\infty\mu + E_0 \left( \frac{\eta_1^2 - \eta_0\mu}{\mu - \eta_0} + \frac{\eta_1^2 + \eta_0\mu}{\mu + \eta_0} \right) + \int_0^\infty [F(\eta, \mu) - F(\eta, -\mu)] E(\eta) d\eta = 0, \quad (5.3)$$

$$E_\infty + E_0 + \int_0^\infty E(\eta) d\eta = 1. \quad (5.4)$$

Extending the function  $E(\eta)$  into the interval  $(-\infty, 0)$  evenly, so that  $E(\eta) = E(-\eta)$ . Then we get the following equation  $F(-\eta, -\mu) = -F(\eta, \mu)$ . We transform the equation (5.3) to the form

$$2E_\infty\mu + E_0 \left( \frac{\eta_1^2 - \eta_0\mu}{\mu - \eta_0} + \frac{\eta_1^2 + \eta_0\mu}{\mu + \eta_0} \right) + \int_{-\infty}^\infty F(\eta, \mu) E(\eta) d\eta = 0, \quad (5.5)$$

We substitute in equation (5.5) the eigenfunctions. We obtain the singular integral equation with the Cauchy kernel on the whole real axis  $-\infty < \mu < +\infty$

$$2E_\infty\mu + E_0 \left[ \frac{\eta_1^2 - \eta_0\mu}{\mu - \eta_0} + \frac{\eta_1^2 + \eta_0\mu}{\mu + \eta_0} \right] + \int_{-\infty}^\infty \frac{\mu\eta - \eta_1^2}{\eta - \mu} E(\eta) d\eta - 2\eta_1^2 w_0 s_0(\alpha) \frac{\Lambda(\mu, \alpha)}{\mu f_0(\mu, \alpha)} = 0. \quad (5.6)$$

We introduce the auxiliary function

$$M(z) = \int_{-\infty}^\infty \frac{z\eta - \eta_1^2}{\eta - z} E(\eta) d\eta \quad (5.7)$$

The function  $M(z)$  is analytic in the complex plane without the cut (the point of integration the whole real axis  $(-\infty, +\infty)$ ). The boundary values of the function  $M(z)$  from above and below by a cut is defined as the limits

$$M^+(\mu) = \lim_{\substack{z \rightarrow \mu, \\ \text{Im } z > 0}} M(z), \quad M^-(\mu) = \lim_{\substack{z \rightarrow \mu, \\ \text{Im } z < 0}} M(z), \quad -\infty < \mu < +\infty.$$

The boundary values of the auxiliary function  $M(z)$  are related by the Sokhotzky formulas

$$M^\pm(\mu) = \pm \pi i (\mu^2 - \eta_1^2) E(\mu) + \int_{-\infty}^\infty \frac{\mu\eta - \eta_1^2}{\eta - \mu} E(\eta) d\eta,$$

where the integral

$$M(\mu) = \int_{-\infty}^\infty \frac{\mu\eta - \eta_1^2}{\eta - \mu} E(\eta) d\eta$$

is understood as singular in terms of the principal value by Cauchy.

The equations follows from Sokhotzky formulas

$$M^+(\mu) - M^-(\mu) = 2\pi i(\mu^2 - \eta_1^2)E(\mu), \quad \mu \in (-\infty, +\infty), \quad (5.8)$$

$$M(\mu) = \frac{M^+(\mu) + M^-(\mu)}{2}, \quad \mu \in (-\infty, +\infty).$$

We transform the singular equation (5.6) with the help of (5.7) and (5.8) to the boundary value problem of Riemann

$$\begin{aligned} & \Lambda^+(\mu, \alpha) \left[ M^+(\mu) + 2E_\infty\mu + E_0 \left( \frac{\eta_1^2 - \eta_0\mu}{\mu - \eta_0} + \frac{\eta_1^2 + \eta_0\mu}{\mu + \eta_0} \right) \right] = \\ & = \Lambda^-(\mu, \alpha) \left[ M^-(\mu) + 2E_\infty\mu + E_0 \left( \frac{\eta_1^2 - \eta_0\mu}{\mu - \eta_0} + \frac{\eta_1^2 + \eta_0\mu}{\mu + \eta_0} \right) \right], \quad \mu \in (-\infty, +\infty). \end{aligned} \quad (5.9)$$

Problem (5.9) has the following solution

$$M(z) = -2E_\infty z - E_0 \left( \frac{\eta_1^2 - \eta_0 z}{z - \eta_0} + \frac{\eta_1^2 + \eta_0 z}{z + \eta_0} \right) + \frac{C_1 z}{\Lambda(z, \alpha)}. \quad (5.10)$$

Let us eliminate the pole of the solution in the infinity. We get that

$$C_1 = 2E_\infty \Lambda_\infty.$$

The amplitude of Debye is in eliminating poles from solutions (5.10) at the points  $\pm\eta_0$ . Since the dispersion function is even then these poles are eliminated one condition

$$E_0 = \frac{C_1 \eta_0}{(\eta_1^2 - \eta_0^2) \Lambda'(\eta_0, \alpha)} = \frac{2E_\infty \Lambda_\infty \eta_0}{(\eta_1^2 - \eta_0^2) \Lambda'(\eta_0, \alpha)}.$$

If we substitute the solution (5.10) in the Sokhotzky formulas (5.8) then we find coefficient of the continuous spectrum

$$E(\mu) = \frac{C_1 \mu}{2\pi i(\mu^2 - \eta_1^2)} \left( \frac{1}{\Lambda^+(\mu) - \Lambda^-(\mu)} \right) = \frac{E_\infty \Lambda_\infty \varepsilon \mu^2 f_0(\mu, \alpha)}{(\varepsilon - i\Omega) \eta_1^2 s_0(\alpha) \Lambda^+(\mu) \Lambda^-(\mu)}.$$

To find the  $E_\infty$  we use the equation (5.4), which we rewrite in view  $E(\eta)$  is even:

$$E_\infty + E_0 + \frac{1}{2} \int_{-\infty}^{\infty} E(\eta) d\eta = 1,$$

or in the explicit form

$$\frac{1}{\Lambda_\infty} + \frac{2\eta_0}{(\eta_1^2 - \eta_0^2) \Lambda'(\eta_0)} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{1}{\Lambda^+(\eta)} - \frac{1}{\Lambda^-(\eta)} \right) \frac{\eta d\eta}{\eta^2 - \eta_1^2} = \frac{1}{\Lambda_\infty E_\infty}. \quad (5.11)$$

The integral from (5.11)

$$J = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{1}{\Lambda^+(\eta)} - \frac{1}{\Lambda^-(\eta)} \right) \frac{\eta d\eta}{\eta^2 - \eta_1^2}.$$

can be calculated analytically. The function

$$\varphi(z) = \frac{z}{\lambda(z)(z^2 - \eta_1^2)},$$

for which  $\varphi(z) = O(z^{-1})$  ( $z \rightarrow \infty$ ) is analytic in the complex plane without the cut with the exception of points  $\pm\eta_1, \pm\eta_0$ . Consequently, this integral is equal to

$$J = \left[ \text{Res}_{\eta_0} + \text{Res}_{-\eta_0} + \text{Res}_{\eta_1} + \text{Res}_{-\eta_1} + \text{Res}_{\infty} \right] \varphi(z).$$

Observing that

$$\text{Res}_{\pm\eta_1} \varphi(z) = \frac{1}{2\Lambda_1}, \Lambda_1 = \Lambda(\eta_1) = 1 - \frac{\varepsilon}{\varepsilon - i\Omega},$$

$$\text{Res}_{\pm\eta_0} \varphi(z) = \frac{\eta_0}{\Lambda'(\eta_0)(\eta_0^2 - \eta_1^2)},$$

we obtain

$$J = \frac{2\eta_0}{\Lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} + \frac{1}{\Lambda_1} - \frac{1}{\Lambda_{\infty}}.$$

Substituting this equality into (5.11) we find that

$$E_{\infty} = \frac{\Lambda_1}{\Lambda_{\infty}}, C_1 = 2\Lambda_1.$$

Thus, the expansion (5.1) and (5.2) are found. The structure of the electric field generally as follows

$$e(x) = \frac{\Lambda_1}{\Lambda_{\infty}} + \frac{2\Lambda_1\eta_0 \exp(-w_0 x/\eta_0)}{\Lambda'(\eta_0, \alpha)(\eta_1^2 - \eta_0^2)} + \frac{\Lambda_1}{w_0\eta_1^2 s_0(\alpha)} \int_0^{\infty} \frac{\eta^2 f_0(\eta, \alpha) \exp(-w_0 x/\eta)}{\Lambda^+(\eta, \alpha) \Lambda^-(\eta, \alpha)} d\eta. \quad (5.12)$$

Recall that the formula (5.12) holds for  $(\Omega, \varepsilon) \in D^+(\alpha)$ . In the case  $(\Omega, \varepsilon) \in D^-(\alpha)$  zero  $\eta_0$  of the dispersion function does not exist. Therefore, we can assume that in this case  $\eta_0 = 0$ . Then the second term in (5.12) vanishes and the formula (5.12) is simplified.

Here is also an explicit expansion of the distribution function  $H(x, \mu)$

$$\begin{aligned} H(x, \mu) = & \frac{\Lambda_1}{w_0\Lambda_{\infty}} \mu + \frac{2\Lambda_1\eta_0}{w_0(\eta_1^2 - \eta_0^2)\Lambda'(\eta_0, \alpha)} \frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} \exp\left(-\frac{w_0 x}{\eta_0}\right) + \\ & + \frac{\Lambda_1}{w_0^2\eta_1^2 s_0(\alpha)} \int_0^{\infty} \exp\left(-\frac{w_0 x}{\eta_0}\right) \frac{F(\eta, \mu) \eta^2 f_0(\eta, \alpha) d\eta}{\Lambda^+(\eta, \alpha) \Lambda^-(\eta, \alpha)}. \end{aligned} \quad (5.13)$$

For the first time the problem of oscillations of electron plasma with a basic equilibrium Fermi–Dirac statistics in a half-space with specular boundary conditions formulated and solved analytically.

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